Topology of certain symplectic conifold transitions of $\mathbb{C}P^1-$bundles

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Abstract

In this paper, we prove the existence of certain symplectic conifold transitions on all $\mathbb{C}P^1-$bundles over symplectic 4–manifolds, which generalizes Smith, Thomas and Yau's examples of symplectic conifold transitions on trivial $\mathbb{C}P^1-$bundles over Kähler surfaces. Our main result is to determine the diffeomorphism types of such symplectic conifold transitions of $\mathbb{C}P^1-$bundles. In particular, this implies that in the case of trivial $\mathbb{C}P^1-$bundles over projective surfaces, Smith, Thomas and Yau's examples of symplectic conifold transitions are diffeomorphic to Kähler 3–folds.

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1 Introduction

In this paper, all manifolds under consideration are closed, oriented and differentiable, unless otherwise stated. By a $\mathbb{C}P^1-$bundle, we always mean the projectivization $\mathbb{P}(E)$ of a complex vector bundle E of rank two.

Symplectic conifold transitions introduced by Smith, Thomas and Yau [15] are symplectic surgeries on symplectic 6–manifolds which collapses embedded Lagrangian 3–spheres and replaces them by symplectic 2–spheres. One sufficient condition to realize such a symplectic surgery is the existence of a nullhomology Lagrangian 3–sphere in the initial symplectic 6–manifold [15, Theorem 2.9]. As a family of typical symplectic 6–manifolds constructed by Thurston [10, Theorem 6.3], $\mathbb{C}P^1-$bundles over symplectic 4–manifolds can be considered as the initial 6–manifolds and one can study the existence of nullhomology Lagrangian 3–spheres in them. In trivial $\mathbb{C}P^1-$bundles over Kähler surfaces, Smith, Thomas and Yau [15] found some

nullhomology Lagrangian 3–spheres and gave examples of symplectic conifold transitions along these Lagrangian 3–spheres which will be called *canonical* in our paper (see Definition 2.1). They pointed out that these examples can produce 3–folds with arbitrarily high second Betti number which are not obviously blowups of smooth 3–folds and it should be possible for them to contain non–Kähler examples. Indeed, Corti and Smith [4] proved that there is such a symplectic conifold transition of the trivial $\mathbb{C}P^1-$bundle over some Enriques surface which is not deformation equivalent to any Kähler 3–fold.

However, our main results in this paper will imply that Smith, Thomas and Yau's examples of symplectic conifold transitions of trivial $\mathbb{C}P^1-$bundles are diffeomorphic to either $\mathbb{C}P^1-$bundles or blowups of $\mathbb{C}P^1-$bundles; in particular, Corti and Smith's examples of symplectic conifold transitions are diffeomorphic to Kähler 3–folds. More generally, we find canonical Lagrangian 3–spheres in all $\mathbb{C}P^1-$bundles over symplectic 4–manifolds (see Lemma 2.2) and prove Theorem 1.1, Corollary 1.3 below.

For simplicity, denote $\overline{\mathbb{C}P^2}$ and S^4 by N_k, $k=1,2$, respectively, where $\overline{\mathbb{C}P^n}$ denotes the complex projective space $\mathbb{C}P^n$ with the opposite orientation. For $k=1,2$, let $\sigma_k \in H_2(N_k)$ and $\sigma_k^* \in H^2(N_k)$ such that σ_1^* is the dual class of the preferred generator σ_1, i.e. $\langle\sigma_1^*, \sigma_1\rangle = 1$, and $\sigma_2 = 0$, $\sigma_2^* = 0$. Denote $[M]$ for the fundamental class of a manifold M. As there are exactly two distinct conifold transitions along a Lagrangian 3–sphere up to diffeomorphism [15], we can state our main results as following.

Theorem 1.1 *Let $\mathbb{P}(E)$ be a symplectic manifold which is the projectivization of a rank two complex vector bundle E over a 4–manifold N. Suppose $\mathbb{P}(E)$ has a canonical Lagrangian 3–sphere. Then the two symplectic conifold transitions of $\mathbb{P}(E)$ along this Lagrangian 3–sphere are diffeomorphic to $\mathbb{P}(E_1)$ and the connected sum $\mathbb{P}(E_2)\sharp\overline{\mathbb{C}P^3}$ respectively, where E_k, $k=1,2$ are the rank two complex bundles over $N\sharp N_k$ with Chern classes satisfying*

$$\begin{aligned} c_1(E_k) &= (c_1(E), -\sigma_k^*) \in H^2(N\sharp N_k) \cong H^2(N)\oplus H^2(N_k); \\ \langle c_2(E_k), [N\sharp N_k]\rangle &= \langle c_2(E), [N]\rangle - 1. \end{aligned}$$

Moreover, if N is symplectic, then the above diffeomorphisms can be chosen to preserve the first Chern classes.

Remark 1.2 *For a 4–manifold N, every pair in $H^2(N)\times H^4(N)$ can be realized as the Chern classes of a rank two complex vector bundles E over N and the isomorphism classes of the bundles E_k in Theorem 1.1 can be completely determined by $c_1(E_k), c_2(E_k)$ [6, Theorem 1.4.20]. Moreover, it is not hard to prove the manifolds $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)\sharp\overline{\mathbb{C}P^3}$ are in different diffeomorphism classes by comparing the cohomology rings.*

The existence of diffeomorphisms preserving the first Chern classes c_1 can be used to define an equivalence relation between symplectic 6–manifolds

[12, 2.1(D)]; for almost complex 6–manifolds, a diffeomorphism preserving c_1 means it preserves the almost complex structures [17, Theorem 9].

As it is well–known that the projectivization of a holomorphic vector bundle over a Kähler manifold admits a Kähler structure [16, Proposition 3.18], we can obtain the following corollary from Theorem 1.1.

Corollary 1.3 *Let $\mathbb{P}(E)$ be the projectivization of a rank two holomorphic vector bundle E over a projective surface N, then the symplectic conifold transitions of $\mathbb{P}(E)$ along a canonical Lagrangian 3–sphere are diffeomorphic to Kähler 3–folds.*

We will finish the proof of Theorem 1.1 and Corollary 1.3 in Section 3.3. In the course of establishing Theorem 1.1, we also compute the topological invariants of $\mathbb{C}P^1-$bundles over simply–connected 4–manifolds in Example 3.1. According to [17] and [16], combining this computation with Theorem 1.1 will give diffeomorphism classification of symplectic conifold transitions of simply–connected $\mathbb{C}P^1-$bundles along canonical Lagrangian 3–spheres.

2 Symplectic conifold transitions on $\mathbb{C}P^1-$bundles

We first recall the definition of conifold transitions [15]. Begin with a Lagrangian embedding $f : S^3 \to X$ in a symplectic 6–manifold X. By the Lagrangian neighborhood theorem [10, Theorem 3.33], the embedding f can extend to a symplectic embedding $f' : T^*_\epsilon S^3 \to X$ with $T^*_\epsilon S^3 \subset T^* S^3$ a neighborhood of the zero section of the cotangent bundle. Define a *conifold transition* along f to be the smooth manifold

$$Y_k := X \backslash f[S^3] \cup_{f' \circ \psi_k} W_k^\epsilon$$

for $k = 1, 2$, where W_k are two small resolutions of the complex singularity $W = \left\{ \sum z_j^2 = 0 \right\} \subset \mathbb{C}^4$ with exceptional set $\mathbb{C}P^1$ over $\{0\} \in W$ and either of W_k is a complex vector bundle over $\mathbb{C}P^1$ with first Chern number -2; fixing coordinates on T^*S^3 as

$$T^*S^3 = \left\{ (u,v) \in \mathbb{R}^4 \times \mathbb{R}^4 | \, |u| = 1, \langle u, v \rangle = 0 \right\},$$

the maps $\psi_k : (W_k \backslash \mathbb{C}P^1 \cong W \backslash \{0\}, \omega_{\mathbb{C}}) \to (T^*S^3 \backslash \{v = 0\}, d(vdu))$ are symplectomorphisms defined in [15, (2.1)] with $\omega_{\mathbb{C}}$ the restriction of the symplectic form $\frac{i}{2} \sum_j dz_j \wedge d\overline{z_j}$ on $\mathbb{C}^4$; the submanifolds $W_k^\epsilon \subset W_k$ are neighborhoods of the exceptional set $\mathbb{C}P^1$ such that $W_k^\epsilon \backslash \mathbb{C}P^1 = \psi_k^{-1}[T^*_\epsilon S^3 \backslash \{v = 0\}]$.

There are more choices in conifold transitions along a Lagrangian 3–sphere than along a Lagrangian embedding $S^3 \to X$, as changing the orientation of the Lagrangian 3–sphere $f[S^3]$ would induce a new Lagrangian

embedding $S^3 \to X$ different from f. However, this change would just swap the diffeomorphism types of the conifold transitions, so there are exactly two distinct conifold transitions Y_k, $k = 1, 2$ along the Lagrangian 3–sphere $f[S^3]$ up to diffeomorphism [15]. It follows from [15, Theorem 2.9] that the two conifold transitions of a symplectic 6–manifold along a nullhomology Lagrangian 3–sphere both admit distinguished symplectic structures. Hence to realize such symplectic conifold transitions on $\mathbb{C}P^1-$bundles, it suffices to find nullhomology Lagrangian 3–spheres.

Inside the product $(\mathbb{C}^2 \times \mathbb{C}P^1, \omega_0 \times \omega_{\mathbb{C}P^1})$ of symplectic manifolds with $\omega_0 = \frac{i}{2}\sum_j dz_j \wedge d\overline{z_j}$ on $\mathbb{C}^2$ and $\omega_{\mathbb{C}P^1}$ the Fubini–Study form on $\mathbb{C}P^1$, a well–known construction [1] of a nullhomology Lagrangian 3–sphere is given by the composition of maps

$$f : S^3 \stackrel{(i,h)}{\to} \mathbb{C}^2 \times \mathbb{C}P^1 \stackrel{\iota \times id_{\mathbb{C}P^1}}{\to} \mathbb{C}^2 \times \mathbb{C}P^1 \tag{1.1}$$

where $i : S^3 \subset \mathbb{C}^2$ is the inclusion of the unit sphere, $h : S^3 \to \mathbb{C}P^1$ is the Hopf map and ι is the complex conjugation on $\mathbb{C}^2$. As the image $f[S^3]$ entirely contains in $B^4(l) \times \mathbb{C}P^1$ with $B^4(l)$ a ball in $\mathbb{C}^2$ of radius $l > 1$, hence finding symplectic embeddings of $B^4(l) \times \mathbb{C}P^1$ in $\mathbb{C}P^1-$bundles would give nullhomology Lagrangian 3–spheres in the bundles. This may lead to the following definition.

Definition 2.1 *Let $\mathbb{P}(E)$ be a symplectic manifold which is a $\mathbb{C}P^1-$bundle over a 4–manifold N. A Lagrangian 3–sphere in $\mathbb{P}(E)$ is called* canonical *if it is the image of the composition of embeddings*

$$S^3 \stackrel{f}{\to} B^4(l) \times \mathbb{C}P^1 \stackrel{\eta}{\to} \mathbb{P}(E)$$

for $l > 1$ where the symplectic embedding η can induce a local trivialization of the bundle $\pi : \mathbb{P}(E) \to N$, i.e. there is a differentiable embedding $k : B^4(l) \to N$ such that

$$\pi^{-1}[k[B^4(l)]] = \eta[B^4(l) \times \mathbb{C}P^1] \stackrel{\eta^{-1}}{\to} B^4(l) \times \mathbb{C}P^1 \stackrel{k \times id_{\mathbb{C}P^1}}{\to} k[B^4(l)] \times \mathbb{C}P^1$$

is a local trivialization of the $\mathbb{C}P^1-$bundle $\mathbb{P}(E)$.

[15] and [4] have shown the existence of canonical Lagrangian 3–spheres in trivial $\mathbb{C}P^1-$bundles over Kähler surfaces. We generalize their result in the following Lemma by Thurston's construction [10, Theorem 6.3] and the construction of Kähler forms on $\mathbb{P}(E)$ [16, Proposition 3.18].

Lemma 2.2 *Let $\mathbb{P}(E)$ be the projectivization of a rank two complex vector bundle E over a symplectic 4–manifold N. Then $\mathbb{P}(E)$ admits a symplectic form such that it has a embedded canonical Lagrangian 3–sphere. Moreover,*

if N is Kähler and E admits a holomorphic structure, then $\mathbb{P}(E)$ admits a Kähler form such that it has a embedded canonical Lagrangian 3–sphere.

Proof. *It suffices to find a symplectic embedding $\eta : B^4(l) \times \mathbb{C}P^1 \to \mathbb{P}(E)$ which can induce a local trivialization with $l > 1$. The keypoint is to note that there exists a system of local trivializations $\left\{(U_j, \phi_j)\right\}_{j=0}^m$ of the bundle $\pi : \mathbb{P}(E) \to N$ and a partition of unity $\rho_j : N \to [0,1]$ subordinating to the open cover $\{U_j\}_{j=0}^m$ of N such that each U_j is contractible and $\rho_0 \equiv 1$ on a nonempty open subset $V \subset U_0$. In fact, this follows easily from [2, Corollary 5.2] .*

For the symplectic case, apply Thurston's construction of the symplectic form to $\mathbb{P}(E)$. Let L^* denote the dual bundle of the tautological line bundle $L = \{(l, v) \in \mathbb{P}(E) \times E \mid v \in l\}$ over $\mathbb{P}(E)$. According to the proof of [10, Theorem 6.3], the first Chern class $c_1(L^*) \in H^2(\mathbb{P}(E))$, the local trivializations $\left\{(U_j, \phi_j)\right\}_{j=0}^m$ and the partition of unity $\rho_j : N \to [0,1]$ can contribute to define a closed 2–form τ on $\mathbb{P}(E)$ such that the restriction of τ on each fiber $\mathbb{C}P^1$ is just $\omega_{\mathbb{C}P^1}$. Moreover, since $\rho_0 \equiv 1$ on V, then the form τ can be chosen such that its restriction on $\pi^{-1}[V]$ is equal to the pullback $\phi_0^* 0 \times \omega_{\mathbb{C}P^1}$ of the form $0 \times \omega_{\mathbb{C}P^1}$ on $V \times \mathbb{C}P^1$. By [10, Theorem 6.3], the 2–form $\tau + \lambda\pi^*\omega_N$ on $\mathbb{P}(E)$ is symplectic for $\lambda > 0$ sufficiently large where ω_N denotes the symplectic form on N. By the Darboux neighborhood theorem, there is always a symplectic embedding $B^4(l) \to (V, \lambda\omega_N)$ with $l > 1$ for λ sufficiently large. So in this case, we have the composition of symplectic embeddings

$$B^4(l) \times \mathbb{C}P^1 \to (V \times \mathbb{C}P^1, \lambda\omega_N \times \omega_{\mathbb{C}P^1}) \xrightarrow{\phi_0^{-1}} (\mathbb{P}(E), \tau + \lambda\pi^*\omega_N)$$

which is the desired embedding.

Now for the Kähler case, assume ω_N is the Kähler form on N and E is holomorphic. Using the system of local trivializations $\left\{(U_j, \varphi_j)\right\}_{j=0}^m$ of E associated to $\left\{(U_j, \phi_j)\right\}_{j=0}^m$ and the partition of unity ρ_j, we can obtain a Hermitian metric h on E such that on the restriction $E|_V$ of E to V, the metric h is induced by the canonical Hermitian metric on $\mathbb{C}^2$ via the projection $E|_V \xrightarrow{\varphi_0} V \times \mathbb{C}^2 \to \mathbb{C}^2$. [16, Proposition 3.18] shows that h induces a Hermitian metric on the bundle L^* and the Chern form ω_E associated to this metric can contribute to obtain a Kähler form $\omega_E + \lambda\pi^*\omega_N$ for $\lambda > 0$ sufficiently large. Replacing τ by ω_E in proof of the symplectic case, we can get the desired symplectic embedding. This completes the proof. ■

3 Topology of symplectic conifold transitions of $\mathbb{C}P^1-$bundles

The aim of this section is to study the topology of symplectic conifold transitions of $\mathbb{C}P^1-$bundles along canonical Lagrangian 3–spheres and prove

Theorem 1.1 and Corollary 1.3. For this purpose, we first recall in Section 3.1 the invariants of simply–connected 6–manifolds with torsion–free homology, and compute the invariants of $\mathbb{C}P^1-$bundles over simply–connected 4–manifolds; then determine in Section 2 the topology of conifold transitions of $B^4(l) \times \mathbb{C}P^1$ along $f[S^3]$, i.e. to establish Lemma 3.2, which is a keypoint to prove Theorem 1.1.

3.1 Invariants of simply–connected 6–manifolds with torsion–free homology

By Wall [17] and Jupp [16], the third Betti number b_3, the integral cohomology ring H^*, the first Pontrjagin class p_1 and the second Whitney–Stiefel class w_2 form a system of invariants, which can distinguish all diffeomorphism classes of simply–connected 6–manifolds with torsion–free homology. As an example, we will compute these invariants for $\mathbb{C}P^1-$bundles over simply–connected 4–manifolds.

Example 3.1 *Let $\pi : \mathbb{P}(E) \to N$ be the projectivization of a rank two complex vector bundle E over a simply–connected 4–manifold. Then $\mathbb{P}(E)$ has a natural orientation which is compatible with that of the base and fibers. By the homotopy exact sequence and Gysin sequence, the 6–manifold $\mathbb{P}(E)$ is a simply–connected with $b_3 = 0$. The cohomology ring and the characteristic classes w_2, p_1 can be computed as follows.*
(i)*By the definition of Chern classes [2, Section 20], we have*

$$H^*(\mathbb{P}(E)) \cong H^*(N)[a]/\left\langle a^2 + \pi^* c_1(E) \cdot a + \pi^* c_2(E) \right\rangle$$

where $a = c_1(L^*)$ with L^* the dual bundle of the tautological line bundle $L = \{(l, v) \in \mathbb{P}(E) \times E \mid v \in l\}$ over $\mathbb{P}(E)$. Let $\{y_i\}$ be a basis of the free $\mathbb{Z}-$module $H^2(N)$ and then $\{a, \pi^* y_i\}$ forms a basis of $H^2(\mathbb{P}(E))$. By the relations $a^2 + \pi^* c_1(E) \cdot a + \pi^* c_2(E) = 0$ and $\langle [N]^* \cup a, [\mathbb{P}(E)] \rangle = 1$ with $[N]^* \in H^4(N)$ satisfying $\langle [N]^*, [N] \rangle = 1$, we can obtain

$$\begin{aligned}
\left\langle a^3, [\mathbb{P}(E)] \right\rangle &= \left\langle c_1(E)^2 - c_2(E), [N] \right\rangle; \\
\left\langle a^2 \cup \pi^* y_i, [\mathbb{P}(E)] \right\rangle &= -\left\langle c_1(E) y_i, [N] \right\rangle; \\
\left\langle a \cup \pi^* y_i \cup \pi^* y_j, [\mathbb{P}(E)] \right\rangle &= \left\langle y_i y_j, [N] \right\rangle; \\
\left\langle \pi^* y_i \cup \pi^* y_j \cup \pi^* y_k, [\mathbb{P}(E)] \right\rangle &= 0.
\end{aligned}$$

(ii)*As the tautological line bundle L is a subbundle of the pullback $\pi^* E$ and a Hermitian metric on E pulls back to a Hermitian metric on $\pi^* E$, we have a splitting $\pi^* E = L \oplus L^\perp$ where $L^\perp$ is the orthogonal complement bundle of L and hence*

$$\begin{aligned}
T\mathbb{P}(E) &\cong \pi^* TN \oplus Hom_{\mathbb{C}}(L, L^\perp) \text{ [11, Theorem 14.10]}; \\
Hom_{\mathbb{C}}(L, L^\perp) \oplus \varepsilon^1_{\mathbb{C}} &\cong L^* \otimes \pi^* E
\end{aligned}$$

with $\varepsilon^1_{\mathbb{C}}$ the trivial complex line bundle. These isomorphisms, together with the relations $a^2 + \pi^* c_1(E) \cdot a + \pi^* c_2(E) = 0$, $p_1 = {c_1}^2 - 2c_2$ and

$$c_1(L_1 \otimes L_2) = 2c_1(L_1) + c_1(L_2); c_2(L_1 \otimes L_2) = c_1(L_1)^2 + c_1(L_1)c_1(L_2) + c_2(L_2)$$

with L_i a complex vector bundle of rank $i = 1, 2$ [11, Problem 16-B], imply

$$\begin{aligned} w_2(T\mathbb{P}(E)) &\equiv \pi^*(w_2(TN) + w_2(E)); \\ p_1(T\mathbb{P}(E)) &= \pi^*(p_1(TN) + c_1(E)^2 - 4c_2(E)). \end{aligned}$$

Thus we have

$$\begin{aligned} \left\langle a^2 \cup w_2(T\mathbb{P}(E)), [\mathbb{P}(E)] \right\rangle &= \left\langle w_2(E) \cup (w_2(E) + w_2(TN)), [\mathbb{P}(E)] \right\rangle; \\ \left\langle a \cup \pi^* y_i \cup w_2(T\mathbb{P}(E)), [\mathbb{P}(E)] \right\rangle &= \left\langle y_i \cup (w_2(E) + w_2(TN)), [\mathbb{P}(E)] \right\rangle; \\ \left\langle \pi^* y_i \cup \pi^* y_j \cup w_2(T\mathbb{P}(E)), [\mathbb{P}(E)] \right\rangle &= 0. \\ \left\langle a \cup p_1(T\mathbb{P}(E)), [\mathbb{P}(E)] \right\rangle &= 3\sigma(N) + \left\langle c_1(E)^2 - 4c_2(E), [N] \right\rangle \\ \left\langle \pi^* y_i \cup p_1(T\mathbb{P}(E)), [\mathbb{P}(E)] \right\rangle &= 0 \end{aligned}$$

where $\sigma(N)$ is the signature of the 4–manifold N [11, SIGNATURE THEOREM 19.4].

3.2 Topology of conifold transitions of $B^4(l) \times \mathbb{C}P^1$ along $f[S^3]$

It is easy to see that the definition of conifold transitions can extend to symplectic manifolds with boundaries. In this subsection we will prove Lemma 3.2, determining the topology of Y_k, $k = 1, 2$, which denote the two conifold transitions of $B^4(l) \times \mathbb{C}P^1$ along the Lagrangian embedding $f : S^3 \to B^4(l) \times \mathbb{C}P^1$ in (1.1).

As in Section 1, denote $\overline{\mathbb{C}P^2}$ and S^4 by N_k, $k = 1, 2$, respectively. Let $\sigma_k \in H_2(N_k)$ and $\sigma_k^* \in H^2(N_k)$ such that σ_1^* is the dual class of the preferred generator σ_1 and $\sigma_2 = 0$, $\sigma_2^* = 0$. As $\partial Y_k = \partial B^4(l) \times \mathbb{C}P^1$, the lemma can be stated as following.

Lemma 3.2 *Let id_∂ denote the identity map of $\partial Y_k = \partial B^4(l) \times \mathbb{C}P^1$. Then there are two diffeomorphisms*

$$\begin{aligned} \phi_1 &: B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_1 \to \mathbb{P}(E_1'); \\ \phi_2 &: B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_2 \to \mathbb{P}(E_2') \sharp \overline{\mathbb{C}P^3} \end{aligned}$$

such that the restriction of ϕ_k on $B^4(l) \times \mathbb{C}P^1$ can induce a local trivialization of the bundle $\mathbb{P}(E_k')$ for $k = 1, 2$, where E_k' is the rank two complex bundle over N_k with $c_1(E_k') = -\sigma_k^*$ and $c_2(E_k') = -1$.

To show this lemma, it needs to compute the topological invariants of $M_k := B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_k$. As Smith and Thomas [14, Proposition 4.2] have

computed the intersection forms of the conifold transitions of $\mathbb{C}P^2 \times \mathbb{C}P^1$ along a canonical Lagrangian 3–sphere, we will extend their computation to the topological invariants of M_k in Lemma 3.3 and Example 3.4.

The following Lemma will be very useful for the computation of invariants of M_k. Referring to the definition of conifold transitions recalled in Section 2, as we have inclusions of the exceptional set $\mathbb{C}P^1 \subset W_k^\epsilon$ and the set $O \times \mathbb{C}P^1 \subset B^4(l) \times \mathbb{C}P^1 \backslash f[S^3]$ with $O \in B^4(l)$ the original point, let C_k and P_k denote the images of the exceptional set $\mathbb{C}P^1$ and the set $O \times \mathbb{C}P^1$ under the natural inclusions $W_k^\epsilon \to Y_k \to M_k$ and $B^4(l) \times \mathbb{C}P^1 \backslash f[S^3] \to Y_k \to M_k$, respectively.

Lemma 3.3 *Let $\sigma \in H_2(\mathbb{C}P^2)$ be the preferred generator. Then there are two differentiable embeddings $r_k : \mathbb{C}P^2 \sharp N_k \to M_k$, $k = 1, 2$ satisfying the following conditions:*

(i) Under the homomorphism

$$r_{k*} : H_2(\mathbb{C}P^2 \sharp N_k) \cong H_2(\mathbb{C}P^2) \oplus H_2(N_k) \to H_2(M_k),$$

the images of σ and σ_k are the homology classes $[P_k]$ and $\frac{1-(-1)^k}{2} \cdot [C_k]$ in $H_2(M_k)$, respectively.

(ii) The Euler class of the normal bundle of r_k is

$$(-\sigma^*, -\sigma_k^*) \in H^2(\mathbb{C}P^2 \sharp N_k) \cong H^2(\mathbb{C}P^2) \oplus H^2(N_k),$$

where $\sigma^* \in H^2(\mathbb{C}P^2)$ is the dual cohomology class of σ;

(iii) In M_k, the intersection number of the submanifolds $r_k[\mathbb{C}P^2 \sharp N_k]$ and C_k is $(-1)^k$.

To show this lemma, first recall some results in the proof of [15, Theorem 2.9] and [10, Theorem 3.33]. Let

$$\Delta_\epsilon = \{(u, v) \in T_\epsilon^* S^3 | (v_1, v_2, v_3, v_4) = \lambda(-u_2, u_1, -u_4, u_3); \lambda \geq 0\}$$

and fix W_k, $k = 1, 2$ as $W^\pm$ in [15], respectively. [15, Theorem 2.9] finds 4–dimensional submanifolds $\widehat{S}_k \subset W_k^\epsilon$, $k = 1, 2$ such that

(1) $\widehat{S}_1$ is the complex line bundle over the exceptional set $\mathbb{C}P^1 \subset W_1^\epsilon$ with Euler class -1 and $\psi_1^{-1}[\Delta_\epsilon \backslash \{v = 0\}] = \widehat{S}_1 \backslash \mathbb{C}P^1$;
(2) $\widehat{S}_2$ is diffeomorphic to $\mathbb{R}^4$ and $\psi_2^{-1}[\Delta_\epsilon \backslash \{v = 0\}]$ is equal to $\widehat{S}_2$ with a point removed.
(3) The intersection number of $\widehat{S}_k$ and the exceptional set $\mathbb{C}P^1$ in W_k^ϵ is $(-1)^k$.

Considering the symplectic form $d(vdu)$ on T^*S^3 and applying [10, Theorem 3.33] to the Lagrangian embedding f, this defines an embedding $\overline{f}$:

$T^*_\epsilon S^3 \to B^4(l) \times \mathbb{C}P^1$ by $\overline{f}(u,v) = \exp_{f(u)}(-J_{f(u)} \circ df_u \circ \Phi_u(v))$, where J is a compatible almost complex structure on $(B^4(l) \times \mathbb{C}P^1, \omega_0 \times \omega_{\mathbb{C}P^1})$ and $\Phi_u : T^*_u S^3 \to T_u S^3$ is an isomorphism determined by the relation $\omega_0 \times \omega_{\mathbb{C}P^1}(df_u \circ \Phi_u(v), J_{f(u)} \circ df_u(v')) = v(v')$ for $v' \in T_q S^3$.

Proof of Lemma 3.3. As [10, Theorem 3.33] shows that $\overline{f}$ is isotopic to a symplectic embedding which represents a Lagrangian neighborhood of f, thus Y_k is diffeomorphic to $B^4(l) \times \mathbb{C}P^1 \backslash f[S^3] \cup_{\overline{f}\circ\psi_k} W^\epsilon_k$. We claim that the restriction of $\overline{f}$ on $\Delta_\epsilon \backslash \{v = 0\}$ is a diffeomorphism onto the relative complement of a closed neighborhood of

$$O \times \mathbb{C}P^1 \subset R_0 = \left\{(\overline{w}, [w]) \in B^4(l) \times \mathbb{C}P^1 | \, |w| < 1\right\}.$$

If it is true, then combining this claim with the conditions (1), (2), (3) above and the fact that R_0 is the open disc bundle over $O \times \mathbb{C}P^1$ with Euler class 1, it would imply that $R_0 \cup_{\overline{f}\circ\psi_k|\psi_k^{-1}[\Delta_\epsilon\backslash\{v=0\}]} \widehat{S}_k \cong \mathbb{C}P^2 \sharp N_k$ are well–defined differentiable submanifolds of $B^4(l) \times \mathbb{C}P^1 \backslash f[S^3] \cup_{\overline{f}\circ\psi_k} W^\epsilon_k \cong Y_k \subset M_k$ for $k = 1, 2$, respectively, which gives embeddings $r_k : \mathbb{C}P^2 \sharp N_k \hookrightarrow M_k$ satisfying (i)(iii). (ii) would follow from the fact that the restriction of the normal bundle of $R_0 \subset B^4(l) \times \mathbb{C}P^1$ to $O \times \mathbb{C}P^1$ has Euler class -1 and so does the restriction of the normal bundle of $\widehat{S}_1 \subset W^\epsilon_1$ to the exceptional set $\mathbb{C}P^1$.

Now it remains to show our claim. Under the identifications

$$\begin{aligned} TS^3 &= T^*S^3 = \left\{(u,v) \in \mathbb{R}^4 \times \mathbb{R}^4 | \, |u| = 1, \langle u, v\rangle = 0\right\}, \\ \mathbb{R}^4 &= \mathbb{C}^2 : (r_1, r_2, r_3, r_4) \longmapsto (r_1 + ir_2, r_3 + ir_4), \end{aligned}$$

it is easy to see that $v(v') = \omega_0(v, Jv') = \omega_0(\overline{v}, J\overline{v'})$ with $(\overline{v}, \overline{v'})$ the complex conjugate of $(v, v') \in T^*_u S^3 \times T_u S^3$. Thus for any $(u, v) \in \Delta_\epsilon \backslash \{v = 0\}$, we have

$$\begin{aligned} v &= \lambda i u = \lambda\sqrt{-1}u, \lambda > 0; \\ df_u(v) &= (\overline{v}, [v]) = (\overline{v}, [0]) \in T_{(\overline{u},[u])} B^4(l) \times \mathbb{C}P^1 = \mathbb{C}^2 \times \mathbb{C}^2/\mathbb{C}u \end{aligned}$$

and hence $\Phi_u(v) = v$. These relations, together with the fact that the complex structure $J_{f(u)}$ on $T_{f(u)} B^4(l) \times \mathbb{C}P^1$ is induced by the multiplication by $i = \sqrt{-1}$ on $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{C}u$, imply

$$\overline{f}(u, v) = \exp_{(\overline{u},[u])}(-\lambda\overline{u}, [0]) = ((1 - \lambda)\overline{u}, [u]) \in R_0 \backslash O \times \mathbb{C}P^1.$$

This completes the proof. ■

Using Lemma 3.3, we can compute the topological invariants of $M_k = B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_k$ for $k = 1, 2$.

Example 3.4 *We first claim that M_k is a simply–connected 6–manifold with $b_3 = 0$ and $H^2(M_k)$ has a basis consists of z_k and x_k, where z_k is the*

Poincaré dual of the submanifold $R_k = r_k[\mathbb{C}P^2 \sharp N_k] \subset M_k$ *and the definition of* x_k *is contained in the following proof of the claim. Since* M_k *is obtained by surgery along an embedding* $S^3 \times D^3 \hookrightarrow S^2 \times S^4$ *with* C_k *the resulting 2–sphere [3], then* M_k *is simply–connected and there is a cobordism* W_k *between* $S^2 \times S^4$ *and* M_k*, assuming* $j_k : S^2 \times S^4 \hookrightarrow W_k$ *and* $j'_k : M_k \hookrightarrow W_k$ *are the inclusions. From the cohomology exact sequence of the pairs* (W_k, M_k) *and* $(W_k, S^2 \times S^4)$*, it is easy to show that* $H_3(M_k) \cong H^3(M_k)$ *is trivial. Furthermore, consider the exact sequence*

$$0 \to H^2(W_k) \xrightarrow{j_k'^*} H^2(M_k) \xrightarrow{\delta} H^3(W_k, M_k), \tag{3.1}$$

then δz_k *is a generator of* $H^3(W_k, M_k)$ *because the value of* δz_k *on the generator of* $H_3(W_k, M_k)$ *is equal to* $\langle z_k, [C_k]\rangle = (-1)^k$ *by Lemma 3.3 (iii). This, together with the isomorphism* $H^2(W_k) \xrightarrow{j_k^*} H^2(S^2 \times S^4)$ *and the exact sequence (3.1), implies that* $x_k := j_k'^* j_k^{*-1} a$ *and* z_k *form a basis of* $H^2(M_k)$*, where* $a \in H^2(S^2 \times S^4)$ *is the dual class of the preferred generator* $\left[S^2\right]$ *of* $H_2(S^2 \times S^4)$.

(i)The cohomology ring of M_k*: The relations* $j'_{k*}[P_k] = j_{k*}[S^2]$ *and* $\delta x_k = 0$*, together with Lemma 3.3 (i) and the fact that* $\langle x_k, [C_k]\rangle$ *is equal to the value of* $\delta x_k \in H^3(W_k, M_k)$ *on the generator of* $H_3(W_k, M_k)$*, imply*

$$\langle r_k^* x_k, \sigma\rangle = \langle x_k, [P_k]\rangle = \langle a, [S^2]\rangle = 1; \langle r_k^* x_k, \sigma_k\rangle = 0 \tag{3.2}$$

for the basis $\sigma, \sigma_k \in H_2(\mathbb{C}P^2 \sharp N_k) \cong H_2(\mathbb{C}P^2) \oplus H_2(N_k)$*. Let* $e(\nu r_k)$ *denote the Euler class of the normal bundle* νr_k *of* r_k*, then it follows from the values(3.2) and Lemma 3.3 (ii) that*

$$\begin{aligned}
\langle z_k^3, [M_k]\rangle &= \langle z_k^2, [R_k]\rangle = \langle e(\nu r_k)^2, [\mathbb{C}P^2 \sharp N_k]\rangle = \frac{1+(-1)^k}{2};\\
\langle z_k x_k^2, [M_k]\rangle &= \langle x_k^2, [R_k]\rangle = \langle r_k^* x_k^2, [\mathbb{C}P^2 \sharp N_k]\rangle = 1;\\
\langle x_k z_k^2, [M_k]\rangle &= \langle x_k z_k, [R_k]\rangle = \langle r_k^* x_k \cup e(\nu r_k), [\mathbb{C}P^2 \sharp N_k]\rangle = -1;\\
\langle x_k^3, [M_k]\rangle &= \left\langle j_k'^* j_k^{*-1} a^3, [M_k]\right\rangle = 0.
\end{aligned}$$

(ii)The first Pontrjagin class of M_k*: The exact sequence*

$$H_7(W_k) \xrightarrow{\partial} H_6(S^2 \times S^4 \sqcup M_k) \to H_6(W_k),$$

together with the relations $\partial\,[W_k] = [M_k] - [S^2 \times S^4]$, $p_1(M_k) = j_k'^* p_1(W_k)$ *and* $\langle p_1(W_k) \cup j_k'^{*-1} x_k, j_{k*}[S^2 \times S^4]\rangle = \langle p_1(S^2 \times S^4) \cup a, [S^2 \times S^4]\rangle = 0$*, imply*

$$\langle p_1(M_k) x_k, [M_k]\rangle = \langle p_1(W_k) \cup j_k'^{*-1} x_k, j'_{k*}[M_k] - j_{k*}[S^2 \times S^4]\rangle = 0.$$

From the relations $p_1(\nu r_k) = e(\nu r_k)^2$, $\langle p_1(\mathbb{C}P^2 \sharp N_k), \left[\mathbb{C}P^2 \sharp N_k\right]\rangle = 3 \cdot \frac{1+(-1)^k}{2}$ *and* $z_k \cap [M_k] = r_{k*}[\mathbb{C}P^2 \sharp N_k]$*, together with Lemma 3.3 (ii) and the decomposition* $r_k^* T M_k = T(\mathbb{C}P^2 \sharp N_k) \oplus \nu r_k$*, we get*

$$\langle p_1(M_k) z_k, [M_k]\rangle = \langle r_k^* p_1(M_k), [\mathbb{C}P^2 \sharp N_k]\rangle = 2 \times (1 + (-1)^k).$$

(iii)The second Whitney class of M_k: As the value $w_2(S^2\times S^4)=0$ and the isomorphism $j_k^*: H^2(W_k)\to H^2(S^2\times S^4)$ imply $w_2(W_k)=0$, thus

$$w_2(M_k)=j_k'^* w_2(W_k)=0.$$

Now we can prove the Lemma 3.2.

Proof of Lemma 3.2. Denote S^6 and $\overline{\mathbb{C}P^3}$ by Q_1 and Q_2, respectively. By Wall and Jupp's classification of simply–connected 6–manifolds with torsion–free homology [17] [9], comparing the invariants of M_k and $\mathbb{P}(E_k')$ (see Example3.4 and Example3.1), we get two diffeomorphisms

$$\varphi_k : M_i \to \mathbb{P}(E_k')\sharp Q_k,\ k=1,2$$

such that $\varphi_k^* a_k = x_k + \frac{1+(-1)^k}{2}\cdot z_k$ for $k=1,2$, $\varphi_1^*\pi_1^*(-\sigma_1^*) = -z_1$ and $\varphi_2^* z' = z_2$, where

$$a_k \in H^2(\mathbb{P}(E_k')\sharp Q_k)\cong H^2(\mathbb{P}(E_k'))\oplus H^2(Q_k)$$

denote the first Chern classes of the dual bundles of the tautological line bundles over $\mathbb{P}(E_k')$ for $k=1,2$, respectively, $\pi_1 : P(E_1')\to\overline{\mathbb{C}P^2}$ is the bundle projection, and

$$z'\in H^2(\mathbb{P}(E_2')\sharp\overline{\mathbb{C}P^3})\cong H^2(\mathbb{P}(E_2'))\oplus H^2(\overline{\mathbb{C}P^3})$$

is the Poincaré dual of the submanifold $\mathbb{C}P^2\subset\overline{\mathbb{C}P^3}$.

We claim that $\varphi_{k*}[O\times\mathbb{C}P^1]=f_{k*}[\mathbb{C}P^1]$ for the submanifold $O\times\mathbb{C}P^1\subset B^4(l)\times\mathbb{C}P^1\subset M_k$ and embeddings $f_k:\mathbb{C}P^1\to\mathbb{P}(E_k')\sharp Q_k$ representing a fiber of $\mathbb{P}(E_k')$. As the relations $\langle z_k,[O\times\mathbb{C}P^1]\rangle=0$ and $j_{k*}'[P_k]=j_{k*}[S^2]=j_{k*}'[O\times\mathbb{C}P^1]$ imply that $[O\times\mathbb{C}P^1]$ is the dual base of $x_k+\frac{1+(-1)^k}{2}\cdot z_k=\varphi_k^*a_k$ in the basis

$$\left\{x_k+\frac{1+(-1)^k}{2}\cdot z_k, z_k\right\}=\left\{\begin{array}{c}\{\varphi_1^*a_1,\varphi_1^*\pi_1^*(-\sigma_1^*)\}\text{ for }k=1,\\ \{\varphi_2^*a_2,\varphi_2^*z'\}\text{ for }k=2,\end{array}\right.$$

comparing this with the fact that $f_{k*}[\mathbb{C}P^1]$ is the dual base of a_k in the basis $\{a_1,\pi_1^*(-\sigma_1^*)\}$ for $k=1$ and in the basis $\{a_2,z'\}$ for $k=2$, respectively, shows the claim.

Since the claim above implies that $\varphi_k|_{O\times\mathbb{C}P^1}$ is homotopic to f_k, then by [7, THEOREM 1] and the isotopy extension theorem [8, Chapter 8, 1.3. Theorem], there is an isotopy $F_t^k:\mathbb{P}(E_k')\sharp Q_k\to\mathbb{P}(E_k')\sharp Q_k$, $0\le t\le 1$, such that $F_0^k=id$ and $F_1^k\circ\varphi_k|_{O\times\mathbb{C}P^1}=f_k$. Let $\overline{f_k}:B^4(l)\times\mathbb{C}P^1\to\mathbb{P}(E_k')\sharp Q_k$ be an extension of f_k which can induce a local trivialization of the bundle $\mathbb{P}(E_k')$, then $F_1^k\circ\varphi_k|_{B^4(l)\times\mathbb{C}P^1}$ and $\overline{f_k}$ determine two closed tubular neighborhoods of $f_k[\mathbb{C}P^1]$. By the ambient isotopy theorem for closed tubular neighborhoods [8, Chapter 4, Section 6, Exercises 9], there exists an

isotopy $H_t^k : \mathbb{P}(E_k')\sharp Q_k \to \mathbb{P}(E_k')\sharp Q_k$, $0 \le t \le 1$, such that $H_0^k = id$, $H_1^k \circ F_1^k \circ \varphi_k[B^4(l) \times \mathbb{C}P^1] = \overline{f_k}[B^4(l) \times \mathbb{C}P^1]$ and

$$g := \overline{f_k}^{-1} \circ H_1^k \circ F_1^k \circ \varphi_k|_{B^4(l)\times\mathbb{C}P^1} : B^4(l) \times \mathbb{C}P^1 \to B^4(l) \times \mathbb{C}P^1$$

is a $B^4(l)-$bundle isomorphism. As the homotopy group $\pi_2(O(4))$ of the real orthogonal group $O(4)$ is trivial, this implies $g|_{\partial B^4(l)\times\mathbb{C}P^1}$ is isotopic to the identity map of $\partial B^4(l) \times \mathbb{C}P^1$ and then similar to the proof of [8, Chapter 8, 2.3], we can extend g to a self–diffeomorphism ϕ of $M_k = B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_k$ which is identity outside a neighborhood of $B^4(l) \times \mathbb{C}P^1$. Consequently, the restriction of $\phi_k := H_1^k \circ F_1^k \circ \varphi_k \circ \phi^{-1}$ on $B^4(l) \times \mathbb{C}P^1$ is equal to $\overline{f_k}$ and hence $\phi_k, k = 1, 2$, are the desired diffeomorphisms. ■

3.3 Topology of symplectic conifold transitions of $\mathbb{C}P^1-$bundles

The establishment of Lemma 3.2 make it possible to prove Theorem 1.1, which determines the diffeomorphism types of symplectic conifold transitions of $\mathbb{C}P^1-$bundles over 4–manifolds along canonical Lagrangian 3–spheres. In this section, we show this theorem and Corollary 1.3.

Proof of the theorem 1.1. From [15, Theorem 2.9] and the definition of the two symplectic conifold transitions Z_k, $k = 1, 2$ along a canonical Lagrangian embedding $S^3 \xrightarrow{f} B^4(l) \times \mathbb{C}P^1 \xrightarrow{\eta} \mathbb{P}(E)$, we get the identification

$$Z_k = \mathbb{P}(E) \cup_\eta M_k \diagdown (\text{Interior } \eta[B^4(l) \times \mathbb{C}P^1])$$

as almost complex manifolds with $B^4(l) \times \mathbb{C}P^1$ seen as a subset of $M_k = B^4(l) \times \mathbb{C}P^1 \cup_{id_\partial} Y_k$. Denote S^6 and $\overline{\mathbb{C}P^3}$ by Q_1 and Q_2, respectively, and let $E \cup_{\mathbb{C}^2} E_k'$ denote the complex vector bundle over the one point union $N \vee N_k$ obtained by identifying one fiber $\mathbb{C}^2$ of the two bundles E and E_k', respectively. The identity map id of $\mathbb{P}(E)$ and the diffeomorphisms $\phi_k : M_k \to \mathbb{P}(E_k')\,\sharp Q_k$ in Lemma 3.2 contribute to define diffeomorphisms

$$\Psi_k : Z_k \xrightarrow{\cong} \mathbb{P}(E_k)\,\sharp Q_k, k = 1, 2$$

where E_k is the pullback bundle of the bundle $E \cup_{\mathbb{C}^2} E_k'$ under the natural map $N\sharp N_k \to N \vee N_k$. It is very easy to get the Chern class of E_k from the isomorphism $H^2(N \vee N_k) \cong H^2(N\sharp N_k)$, the homomorphism

$$H^4(N \vee N_k) \cong \mathbb{Z} \oplus \mathbb{Z} \to H^4(N\sharp N_k) \cong \mathbb{Z} : (a, b) \longmapsto a + b$$

and the values

$$c_j(E \cup_{\mathbb{C}^2} E_k') = (c_j(E), c_j(E_k)) \in H^{2j}(N) \oplus H^{2j}(N_k) \cong H^{2j}(N \vee N_k)$$

for $j = 1, 2$.

Assume N is symplectic. To prove the diffeomorphisms Ψ_k preserve c_1, consider the commutative diagram

$$\begin{array}{ccc} H^2(\mathbb{P}\,(E_k)\,\sharp Q_k) & \overset{\Psi_k^*}{\to} & H^2(Z_k) \\ \uparrow\cong & & \uparrow\cong \\ H^2(\mathbb{P}\,(E)\cup_{\eta\circ\overline{f_k}^{-1}}\mathbb{P}\,(E'_k)\,\sharp Q_k) & \overset{(id\cup\phi_k)^*}{\to} & H^2(\mathbb{P}\,(E)\cup_\eta M_k) \\ \downarrow & & \downarrow \\ H^2(\mathbb{P}\,(E))\oplus H^2(\mathbb{P}\,(E'_k)\,\sharp Q_k) & \overset{id^*\oplus\phi_k^*}{\to} & H^2(\mathbb{P}\,(E))\oplus H^2(M_k) \end{array} \tag{3.3}$$

where $\overline{f_k}: B^4(l)\times\mathbb{C}P^1\to\mathbb{P}\,(E'_k)\,\sharp Q_k$ is the restriction of ϕ_k as in the proof of Lemma 3.2 and the vertical homomorphisms are induced by the natural inclusions. As the conifold transitions is an almost complex operation preserving the first Chern class [15] [4, Lemma 2], the formula of the first Chern class of a blowup at a point [5, p.608] and $c_1(T\mathbb{P}\,(E)) = 2a+\pi^*(c_1(TN)+c_1(E))$ by Example 3.1(ii), imply that the images of $c_1(T\mathbb{P}\,(E_k)\,\sharp Q_k)$ and $c_1(TZ_k)$ under the vertical composite homomorphisms are

$$\left(2a+\pi^*(c_1(TN)+c_1(E)), 2a_k-(1+(-1)^k)\cdot z'\right), \tag{3.4}$$

$$(2a+\pi^*(c_1(TN)+c_1(E)), 2x_k)\,, \tag{3.5}$$

respectively, with a_k, z' and x_k defined in the proof of Lemma 3.2 and Example 3.4. Since $\phi_k^* a_k = x_k + \frac{1+(-1)^k}{2}\cdot z_k$, $\phi_2^* z' = z_2$ by the proof of Lemma 3.2, then the horizontal homomorphism $id^*\oplus\phi_k^*$ maps the class (3.4) to (3.5) and hence $c_1(TZ_k) = \Psi_k^* c_1(T\mathbb{P}\,(E_k))$ as the vertical homomorphisms in the diagram (3.3) are injective. This completes the proof. ■

Now we turn to show Corollary 1.3.

Proof of Corollary 1.3. As the blowup of a Kähler manifold at a point is also Kähler [16, Proposition 3.24], this Corollary follows easily from Theorem 1.1 and the claim that both E_k over the projective surfaces $N\sharp N_k$ admit holomorphic structures. To prove the claim, it suffices to note Schwarzenberger [13, Theorem 9] showed that a complex vector bundle over a projective surface S admits a holomorphic structure if and only if the first Chern class of the bundle belongs to $H^{1,1}(S)$. As $c_1(E_2) = c_1(E)$ and $c_1(E_1)$ is equal to $c_1(E)$ plus the exceptional divisor $-\sigma_1^*$, so $c_1(E_k)\in H^{1,1}(N\sharp N_k)$ by the Lefschetz theorem on (1,1) classes [16, Theorem 11.30]. This completes the proof. ■

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